Dynamical Symmetry Breaking on a Cylinder in Magnetic Field

A. V. Gamayun $^{(a,c)}$ [1] **and** **E.V. Gorbar** $^{(b,c)}$ [2]

(a) Institute of Physics and Technology of National Technical University, "Kiev Polytechnic Institute" Peremogy Av., 37, 03056 Kiev, Ukraine

(b) Department of Applied Mathematics, University of Western Ontario, London, Ontario N6A 5B7, Canada

(c) Bogolyubov Institute for Theoretical Physics, 03143 Kiev, Ukraine

Abstract

We study dynamical symmetry breaking on a cylinder in external magnetic field parallel to the axis of cylinder when magnetic field affects the dynamics of fermions only through the Aharonov-Bohm phase. We find that unlike other previously studied cases magnetic field in our case counteracts the generation of dynamical fermion mass which decreases with magnetic field. There exists also a purely kinematical contribution to the fermion gap which grows linearly with magnetic field. Remarkably, we find that the total fermion gap, which includes both the dynamical and kinematical contributions, always increases with magnetic field irrespectively the values of coupling constant and the radius of cylinder. Thus, although the dynamical mass is suppressed, external magnetic field does enhance the total fermion gap in the spectrum.

[1]E-mail address: gamayun@bitp.kiev.ua
[2]E-mail address: egorbar@uwo.ca

1 Introduction

It is well known that external magnetic field is a strong catalyst of dynamical symmetry breaking leading to the generation of dynamical fermion mass even at the weakest attractive interaction between fermions [1] (for earlier consideration of dynamical symmetry breaking in a magnetic field, see [2, 3]). The applications of this effect were considered in condensed matter and cosmology (for reviews see [4]). Recently, the idea of magnetic catalysis was used for explanation of the experimentally observed magnetic field driven metal-dielectric phase transition in pyrolytic graphite [5].

As well known, carbon nanotubes (see, e.g., [6]) are essentially graphite sheets wrapped on a cylinder. Experimentally produced carbon nanotubes have small radii and their long wave length excitations are adequately described by a (1+1)-dimensional effective quantum field theory whose spatial content is a $\mathbf{R}^1$ space because only the lowest mode for fermions on the circle is retained. Nonetheless, the cylinder geometry of carbon nanotubes presents an interesting setup from the viewpoint of dynamical symmetry breaking in external constant magnetic field in a $(2+1)$-dimensional spacetime with nontrivial topology. We study this problem in the present paper. It is clear that two directions of magnetic field are distinguished. The first one is when magnetic field is perpendicular to the axis of cylinder and the second when it is parallel. Obviously, since the normal projection of magnetic field varies with angle, the first case presents an inhomogeneous problem whose solution is difficult to find. The case where magnetic field is parallel to the axis of cylinder is much more tractable and was already studied in some detail in [7]. Classically, magnetic field parallel to the axis of cylinder does not affect at all the motion of charged particles on the surface of cylinder. However, this is not true in quantum mechanics where the presence of magnetic field leads to the appearance of the Aharonov–Bohm phase [8]. We would like to add also that, in general, one can relax the requirement that external magnetic field is constant. It is enough to require only that the normal component of magnetic field be equal to zero on the surface of cylinder. Magnetic field through the transverse section of cylinder can be arbitrary and what matters is only the total magnetic field flux through the transverse section of cylinder.

To study dynamical symmetry breaking on a cylinder in parallel magnetic field, we consider the following Gross-Neveu type model [9] with N flavors:

$$\mathcal{L} = \sum_{k=1}^{N} \bar{\psi}_k i\gamma^\mu D_\mu \psi_k + \frac{G}{2N}(\bar{\psi}\psi)^2, \tag{1}$$

where $D_\mu = \partial_\mu - ieA_\mu^{ext}$ and $\bar{\psi}\psi = \sum_{i=1}^{N} \bar{\psi}_i\psi_i$, N the number of flavors. According to [10], we consider the four-component spinors corresponding to a four-dimensional (reducible) representation of Dirac's matrices

$$\gamma^0 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}. \tag{2}$$

The Lagrangian (1) is invariant under the discrete transformation $\psi \to e^{i\frac{\pi}{2}\gamma^5}\psi$, where

$$\gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Obviously, the mass term breaks this symmetry since

$$\bar{\psi}\psi \to -\bar{\psi}\psi.$$

We consider the model with breaking of discrete symmetry in order to avoid strong quantum fluctuations due to massless Nambu–Goldstone fields which, if present, would destroy the mean-field solution that we find.

2 Dynamical mass generation

Introducing an auxiliary σ field, the Lagrangian (1) can be equivalently represented in the following way:

$$\mathcal{L} = \sum_{k=1}^{N}(\bar{\psi}_k i\gamma^\mu D_\mu \psi_k - \sigma\bar{\psi}_k\psi_k) - \frac{N}{2G}\sigma^2 \,. \tag{3}$$

Integrating over the fermion fields, we obtain the following effective action for the σ field in the $\frac{1}{N}$ approximation:

$$\Gamma(\sigma) = -\frac{N}{2G}\int \sigma^2 d^3x - iNTrLn(i\gamma^\mu D_\mu - \sigma). \tag{4}$$

If we choose the x axis along the axis of cylinder and the y axis along the circumference, then, mathematically, cylinder is a $\mathbf{R}^1 \times \mathbf{S}^1$ space or compactified plane in the y direction, i.e. $y \sim y + 2\pi R$, where R is the radius of cylinder. Obviously, $\mathbf{R}^1 \times \mathbf{S}^1$ is a multiply-connected space and, according to Hosotani [11], the nonzero component A_y of gauge field cannot be gauged away unlike the case of a simply connected space. Equivalently, if we consider our space $\mathbf{R}^1 \times \mathbf{S}^1$ as a real cylinder in embedding 3D space, then the constant A_y component is related to the flux of constant magnetic field $\Phi = 2\pi R A_y = \pi R^2 B$ through the transverse section of cylinder [12]. Further,

$$Tr\ln(i\gamma^\mu D_\mu - \sigma) = -Tr\int_0^\sigma ds\frac{1}{i\gamma^\mu D_\mu - s} = 4\int d^3x \int_0^\sigma sdsG(\mathbf{x},\mathbf{x};s) \tag{5}$$

and, consequently, the effective potential is equal to

$$V(\sigma) = \frac{N\sigma^2}{2G} + 4Ni\int_0^\sigma sdsG(\mathbf{x},\mathbf{x};s), \tag{6}$$

where $G(\mathbf{x},\mathbf{x}';s)$ is Green's function which satisfies the following equation:

$$(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - (\frac{\partial}{\partial y} - ieA)^2 + s^2)G(\mathbf{x},\mathbf{x}';s) = \delta(x-x'), \tag{7}$$

where $A_y = \frac{RB}{2}$. Obviously, in Euclidean space,

$$G(\mathbf{x},\mathbf{x};s) = -i\int \frac{d^2p}{(2\pi)^2 L}\sum_n \frac{1}{p^2 + (\frac{2\pi}{L})^2(n-\phi_\parallel)^2 + s^2}\,, \tag{8}$$

where $L = 2\pi R$, $\phi_{\|} = \Phi/\Phi_0 = \frac{eA_yL}{2\pi}$, and $\Phi_0 = 2\pi/e$ is the elementary magnetic field flux. Since the sum in (8) is over all integer n, it suffices to consider $\phi_{\|}$ on the interval $[0, \frac{1}{2}]$. To evaluate the sum in (8), we transform it into an integral in complex plane ω

$$\sum_n \frac{1}{(p^2+s^2)(L/2\pi)^2+(n-\phi_{\|})^2} = \frac{1}{2\pi i}\int_C \frac{d\omega}{1-e^{2\pi(\omega+i\phi_{\|})}}\frac{-2\pi}{(p^2+s^2)(L/2\pi)^2-\omega^2}\,, \tag{9}$$

where the contour C runs around poles of the function $(1-e^{2\pi(\omega+i\phi_{\|})})^{-1}$. Then we can deform contour and present the last integral as sum over two residues at $\omega = \pm L/2\pi(p^2+s^2)^{1/2}$. As result, we have

$$G(\mathbf{x},\mathbf{x};s) = -i\frac{1}{4\pi}\int \frac{pdp}{(p^2+s^2)^{1/2}}\frac{\sinh(L(p^2+s^2)^{1/2})}{\cosh(L(p^2+s^2)^{1/2})-\cos(2\pi\phi_{\|})} = \tag{10}$$

$$-\frac{i}{4\pi L}\ln\frac{\cosh(L(\Lambda^2+s^2)^{1/2})-\cos(2\pi\phi_{\|})}{\cosh(Ls)-\cos(2\pi\phi_{\|})}\,.$$

Finally, the effective potential is

$$V(\sigma) = \frac{N\sigma^2}{2G} - \frac{N}{\pi L}\int_0^\sigma \ln\frac{\cosh(L(\Lambda^2+s^2)^{1/2})-\cos(2\pi\phi_{\|})}{\cosh(Ls)-\cos(2\pi\phi_{\|})}sds, \tag{11}$$

where Λ is cut-off. The gap equation which follows from $\frac{dV(\sigma)}{d\sigma} = 0$ is

$$\frac{\sigma}{G} = \frac{\sigma}{\pi L}\ln\frac{\cosh(L(\Lambda^2+\sigma^2)^{1/2})-\cos(2\pi\phi_{\|})}{\cosh(L\sigma)-\cos(2\pi\phi_{\|})}\,. \tag{12}$$

Consequently, the nontrivial solution for $\Lambda \to \infty$ is

$$\sigma = \frac{1}{L}\cosh^{-1}\left(\frac{e^{L(\Lambda-\frac{\pi}{G})}}{2} + \cos(2\pi\phi_{\|})\right)\,, \tag{13}$$

which obviously decreases with $\phi_{\|}$ (see Fig.1). We would like to note that our effective potential and the solution of gap equation agree with the corresponding results obtained in [7]. By using (11), it is not difficult to show that the nontrivial solution always has lower energy than the trivial solution $\sigma = 0$.

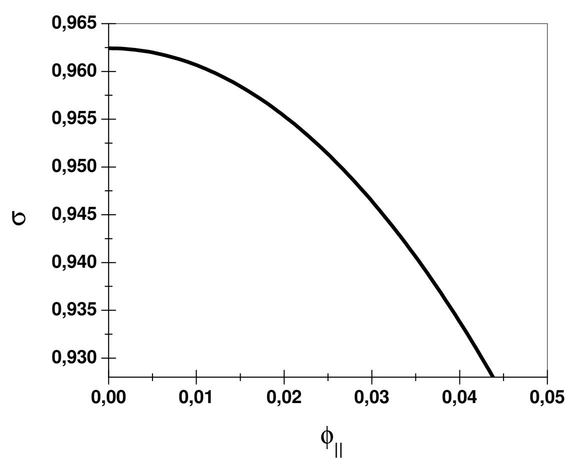

FIG.1. The dependence of σ on $\phi_{\|}$.

This result corresponds to an unusual situation where external constant magnetic field inhibits rather than assists dynamical symmetry breaking. According to [1], constant magnetic field in infinite flat space leads to dimensional reduction by two units $D+1 \to (D-2)+1$ in the infrared region for fermions that strongly assists dynamical symmetry breaking. Magnetic field in our case does not influence the spatial motion of fermions at all, therefore, the dimensional reduction is absent. The only dynamical effect that magnetic field has in our case is through the appearence of the Aharonov–Bohm phase. Since this phase increases energy of fermions, we obtain that magnetic field in our case counteracts dynamical symmetry breaking unlike all other known cases. Remarkably, this result is consistent with a more general idea that external magnetic field enhances the fermion gap in the spectrum. The point is that there is an additional purely kinematic contribution $\frac{2\pi\phi_{\|}}{L}$ (see (8)) due to the Aharonov–Bohm phase to the total fermion gap and as we show below the total gap $\sigma_{tot}^2 = \sigma^2 + (\frac{2\pi\phi_{\|}}{L})^2$ increases with $\phi_{\|}$. To prove this, let us consider the function

$$f(x) = (\cosh^{-1}(a+\cos x))^2 + x^2,$$

where $x = 2\pi\phi_{\|}$ takes values on the interval $[0,\,\pi]$ and $\xi = a + \cos x > 1$, otherwise, the gap equation (12) does not have a nontrivial solution. The derivative of this function with respect to x is equal to

$$\frac{df(x)}{dx} = 2x\left(1 - \frac{\sin x}{x}\frac{\ln(\xi + \sqrt{\xi^2-1})}{\sqrt{\xi^2-1}}\right).$$

Further, taking into account that

$$\frac{\sin x}{x} \le 1 \qquad \text{and} \qquad \frac{\ln(\xi+\sqrt{\xi^2-1})}{\sqrt{\xi^2-1}} < 1\,,$$

we find that this derivative is positive

$$\frac{df(x)}{dx} > 0\,.$$

Thus, although the dynamically generated fermion mass decreases with $\phi_{\|}$, the total fermion gap increases with $\phi_{\|}$. Graphically, the dependence of σ_{tot} on $\phi_{\|}$ is depicted in Fig.2.

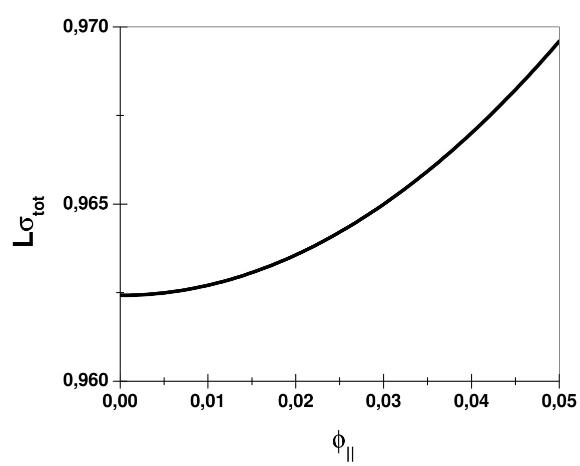

FIG.2. The dependence of σ_{tot} on $\phi_{\|}$.

Finally, let us find the critical coupling constant that separates the symmetrical phase from the phase with broken symmetry. As follows from (13), the critical coupling constant is

$$G_c(\phi_{\|}) = \frac{\pi L}{\Lambda L - \ln[2(1-\cos(2\pi\phi_{\|}))]}\,. \tag{14}$$

Once again we see that external magnetic field counteracts dynamical symmetry breaking since as follows from (14) the critical coupling constant increases with $\phi_{\|}$ that means that we need stronger attraction in order to break symmetry. For $\phi_{\|} \to 0$, the critical coupling constant goes to zero $G_c \to 0$, i.e. symmetry is always broken in this case.By using (13) one can show that the phase transition with respect to the magnetic field flux is the mean field second order phase transition.

3 Conclusions

In this paper we have investigated dynamical symmetry breaking on a cylinder in external magnetic field parallel to the axis of cylinder. This problem may be relevant for certain condensed matter systems. In fact, we were inspired by the geometry of carbon nanotubes. On the other hand, our problem can be considered as the problem of dynamical symmetry breaking in a multiply-connected space, where gauge field has a nonzero constant vacuum expectation value [11], which cannot be gauged away unlike the case of a simply connected space.

By studying a relatively simple Gross–Neveu type model on a cylinder in external magnetic field, we find that unlike all other known cases external magnetic field counteracts the generation of dynamical fermion mass, i.e. larger magnetic fields correspond to smaller dynamical fermion masses. There exists also an additional purely kinematic contribution to the fermion gap in this problem, which increases with magnetic field. Remarkably, we find that the total fermion gap, which includes both the dynamical and kinematical contributions, always increases with magnetic field irrespectively the values of coupling constant and the radius of cylinder. Thus, although our analysis shows that external magnetic field in spaces with nontrivial topology does not always assist dynamical symmetry breaking unlike the case of flat spaces with trivial topology [1], our results are consistent with a more general idea that external magnetic field increases the fermion gap in the spectrum.

We would like to note that one can gauge away the vector potential in our problem but in this case the Aharonov–Bohm phase will reveal itself in the boundary conditions for fermions on the circle. The cases of periodic and antiperiodic boundary conditions for fermion fields were considered in [13], where dynamical chiral symmetry breaking was studied in a spacetime $\mathbf{R^3} \times \mathbf{S^1}$ in external magnetic field.[3] It is clear that the Aharonov–Bohm phase defines arbitrary boundary conditions for fermions on the circle and allows to interpolate smoothly between the periodic and antiperiodic boundary conditions. It is easy to check that our results on dynamical symmetry breaking are consistent with the results of [13] (as well as [14]), where it was found that although for periodic boundary conditions (when the Aharonov–Bohm phase is zero in our setup) a dynamical mass is generated at the weakest attractive interaction between fermions, the effect of antiperiodic boundary conditions (when the Aharonov–Bohm phase

[3] The role of periodic and antiperiodic boundary conditions of fermion fields in the dynamical symmetry breaking in flat spaces with more general topology but without external magnetic field was considered in [14].

attains its maximal possible value) is to counteract the dynamical chiral symmetry breaking. Since magnetic field considered in [13] is perpendicular to the spatial plane, unlike our case it cannot influence boundary conditions for fermions on the circle. The paper [15] studied dynamical chiral symmetry breaking in QED in a spacetime with the topology $\mathbf{R}^3 \times \mathbf{S}^1$ without external magnetic field. In this case the Aharonov-Bohm phase for fermions appears due to nonzero constant vacuum expectation value of the electromagnetic vector potential that cannot be gauged away in a multiply-connected spacetime. The Aharonov-Bohm phase in this problem is not an external parameter like in our problem but a dynamical variable that should be determined from the requirement of the minimum of energy. Interestingly, it was found in [15] that the state with the lowest energy corresponds to fermions with antiperiodic boundary conditions. This disfavors dynamical fermion mass generation, however, according to the results of our paper, the total fermion gap which includes also the purely kinematic contribution due to the Aharonov-Bohm phase is still larger than the gap in the case of zero Aharonov-Bohm phase.

Finally, we would like to note that a $(2+1)$-dimensional free fermion model was investigated in [16], where a nonzero chiral symmetry breaking condensate was found due to the Aharonov–Bohm phase related to the singular magnetic vortex. In our opinion, it would be very interesting to study dynamical symmetry breaking in this model by adding some interaction. Unfortunately, unlike the problem that we investigated in this paper the problem considered in [16] is inhomogeneous and such a study would be very difficult to perform.

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